Nonlinear Dirac soliton in an external field

1995 J. Phys. A: Math. Gen. 28 1413

(http://iopscience.iop.org/0305-4470/28/5/025)

J. Phys. A: Math. Gen. 28 (1995) 1413–1424. Printed in the UK

Nonlinear Dirac soliton in an external field

Y Nogami†§, F M Toyama‡ and Z Zhao†

† *Department of Physics and Astronomy, McMaster University, Hamilton,* Ontario, Canada L8S 4M1

‡ Department of Communication and Information Sciences, Kyoto Sangyo University, Kyoto 603, Japan

Received 20 September 1994

Abstract. The behaviour of the nonlinear Dirac soliton in an external potential in $1 + 1$ dimensions is examined by means of a collective variable ansatz and also by solving the nonlinear Dirac equation numerically. When the potential is linear with respect to coordinate x, the motion of the soliton centroid is found to be consistent with the classical relativistic equation of motion for a point particle. For a general potential there is a deviation from the behaviour of the corresponding classical point particle. This deviation can be interpreted as being caused by the finite size of the soliton.

1. Introduction

It is known that the nonlinear Schrödinger (NLS) soliton in $1 + 1$ dimensions, placed in an external potential field $V(x)$ which is linear or quadratic with respect to coordinate x, moves exactly like a classical point particle [1, 2]. The soliton centroid (centre of mass) obeys Newton's equation of motion in a specified external potential. This can be shown analytically on the basis of an analogue of Ehrenfest's theorem of quantum mechanics. Having seen this it is natural to ask the following question: Does the nonlinear Dirac (NLD) soliton in an external potential also behave like a classical relativistic particle? In other words, does the centroid of the NLD soliton obey the classical relativistic equation of motion for a point particle? The purpose of this paper is to find an answer to this question. *Throughout this paper we work in 1 + 1 dimensions.*

We attempted to find the answer analytically but without success. One can see why it is difficult to do so if one recalls the fact that there is no straightforward analogue of Ehrenfest's theorem for the Dirac equation in quantum mechanics. Unlike in the Schrödinger case, it is difficult to relate $\mathrm{d}\langle x\rangle/\mathrm{d}t$ to $\langle p\rangle$ in the relativistic case where $\langle\,\rangle$ means the expectation value. In this paper we examine the problem in two ways: (i) by means of a collective variable ansatz and (ii) by solving the NLD equation numerically. (i) is analytical but approximate; (ii) is essentially exact but there is a limit to the numerical accuracy. The results of the two methods complement each other. When $V(x)$ is linear in x, our results strongly indicate that the motion of the soliton centroid is consistent with the classical relativistic equation of motion for a point particle. For $V(x)$ of general form, we find a small deviation from the behaviour of the corresponding point particle. This deviation can be interpreted as being caused by the finite size of the soliton.

§ Correspondence address (e-mail: nogami@mcmaster.ca).

0305-4470/95/051413+12$19.50

Let us add a remark about the usage of the word 'soliton'. A soliton is a special solitary wave such that, when it collides with another soliton, each of the solitons comes out exactly in the same shape as it had before the collision. The NLS soliton is a well-known example. Strictly speaking, the NLD solitary wave that we are going to consider is not a soliton. Its collision with another NLD soliton may lead to processes which resemble compound nuclear reactions [3]. Nevertheless let us call it the NLD soliton for brevity. We do not consider soliton–soliton collisions in this paper. Rather we focus on an isolated soliton in an external field.

In section 2 we summarize the one-soliton solution of the NLD equation with no external potential. In section 3 we examine a collective variable ansatz which leads to an equation of motion for the centroid of the NLD soliton in an external potential. Section 4 deals with an alternative form of the NLD equation which we also use in our numerical analysis. In section 5 we examine the behaviour of the NLD soliton in an external potential by solving the NLD equation numerically. The results are discussed in section 6.

2. One-soliton solution of the NLD equation

The NLD equation that we consider in this section is

$$\mathrm{i}\partial_t\psi = [-\mathrm{i}\alpha\partial_x + \beta m - g(\psi^\dagger\beta\psi)\beta]\psi \tag{2.1}$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\psi^\dagger$ is the Hermitian adjoint of ψ, $m(>0)$ is a 'mass' parameter and g is a dimensionless coupling constant which we assume to be positive. For the 2×2 Dirac matrices α and β, we take $\alpha = \sigma_y$ and $\beta = \sigma_z$ where the σ's are the usual Pauli matrices. Throughout this paper we use units such that $c = \hbar = 1$. For the normalization of ψ, it is understood that

$$\int_{-\infty}^{\infty}\rho(x,t)\,\mathrm{d}x \equiv \int_{-\infty}^{\infty}\psi^\dagger(x,t)\psi(x,t)\,\mathrm{d}x = 1 \tag{2.2}$$

which is consistent with (2.1). The nonlinear term of (2.1) is the Lorentz scalar type. There are other types of nonlinearity, vector and pseudoscalar, or their linear combinations [4], but we focus on the NLD soliton with the scalar nonlinearity as a typical case.

Let us first examine the soliton which is at rest at the origin. In this case we can assume that

$$\psi(x,t) = \phi(x)\mathrm{e}^{-\mathrm{i}\epsilon t}. \tag{2.3}$$

Then (2.1) is reduced to

$$\epsilon\phi = [-\mathrm{i}\alpha\,\mathrm{d}_x + \beta m - g(\phi^\dagger\beta\phi)\beta]\phi \tag{2.4}$$

where $\mathrm{d}_x = \mathrm{d}/\mathrm{d}x$. The eigenvalue ϵ and the wavefunction $\phi(x)$ are given by

$$\epsilon = m[1 + (g^2/4)]^{-1/2} \tag{2.5}$$

$$\phi(x) = \frac{(\kappa\epsilon)^{1/2}}{m + \epsilon\cosh(2\kappa x)}\begin{bmatrix}(m+\epsilon)^{1/2}\cosh(\kappa x)\\ -(m-\epsilon)^{1/2}\sinh(\kappa x)\end{bmatrix} \tag{2.6}$$

where $\kappa = (m^2 - \epsilon^2)^{1/2} = g\epsilon/2$ [3,4]. The energy of the soliton is given by

$$M = \int_{-\infty}^{\infty} \mathrm{d}x \left[\phi^\dagger(-\mathrm{i}\alpha\,\mathrm{d}_x + \beta m)\phi - \frac{g}{2}(\phi^\dagger\beta\phi)^2\right]$$
$$= \frac{m}{g}\ln\left(\frac{m+\kappa}{m-\kappa}\right). \tag{2.7}$$

This M, which we interpret as the rest mass of the soliton, should not be confused with the mass parameter m that appears in the Dirac equation.

The wavefunction $\psi(x, t; v)$ for the soliton moving with a constant speed v can be obtained by applying the Lorentz transformation to the soliton at rest [3]:

$$\psi(x, t; v) = L(\gamma)\phi(x')\mathrm{e}^{-\mathrm{i}\epsilon t'} \tag{2.8}$$

$$L(\gamma) = 2^{-1/2}(\gamma+1)^{1/2}\left(1 + \frac{v\gamma}{\gamma+1}\sigma_y\right) \tag{2.9}$$

$$\gamma = (1-v^2)^{-1/2} \qquad x' = \gamma(x - vt) \qquad t' = \gamma(t - vx) \tag{2.10}$$

where $\phi(x')$ is the $\phi(x)$ with x replaced by x' and σ_y is one of the Pauli matrices. Note that Alvarez and Carreras [3] used $\alpha = \sigma_x$ and $\beta = \sigma_z$. If we denote their two-component wavefunction by ψ_A, it is related to our ψ by

$$\psi_\mathrm{A} = \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\psi. \tag{2.11}$$

The energy and momentum carried by the soliton moving with speed v are given by

$$E = \gamma M = (M^2 + p^2)^{1/2} \qquad p = vE. \tag{2.12}$$

These are exactly the same as the energy and momentum of a point particle of rest mass M and speed v.

3. Collective variable ansatz

We now examine the behaviour of the NLD soliton placed in an external potential. The NLD equation is

$$\mathrm{i}\partial_t\psi = [-\mathrm{i}\alpha\partial_x + \beta m - g(\psi^\dagger\beta\psi)\beta + V(x)]\psi. \tag{3.1}$$

The external potential $V(x)$ is the zeroth component of a Lorentz vector, like the Coulomb potential. Rather than trying to solve this equation, we introduce a collective variable ansatz for the ψ of the soliton. Similar methods have been used for the NLS soliton [5] and also for the sine–Gordan soliton [6]. Earlier references can be traced through [5,6].

Our ansatz is to assume that the ψ for the NLD soliton, whose centroid is at $x = \xi(t)$, is given by $\psi(x, t; v)$ of (2.8) with the understanding that $v = \dot{\xi} = \mathrm{d}\xi/\mathrm{d}t$ and

$$\gamma = (1-\dot{\xi}^2)^{-1/2} \qquad x' = \gamma(x - \xi) \qquad t' = \gamma(t - \dot{\xi}x). \tag{3.2}$$

The ansatz implies that the structure of the soliton placed in an external potential remains essentially the same as that of the free soliton. The transformation $(x, t) \to (x', t')$, however, is no longer a Lorentz transformation because the speed $\dot{\xi}$ is not a constant. The inverse transformation is complicated.

The energy of the soliton is given by

$$E = \int_{-\infty}^{\infty} \mathrm{d}x \left[\psi^\dagger(-\mathrm{i}\alpha\partial_x + \beta m)\psi - \frac{g}{2}(\psi^\dagger\beta\psi)^2 + V(x)\psi^\dagger\psi\right]. \tag{3.3}$$

Substituting $\psi(x, t; \dot{\xi})$ into (3.3), we obtain

$$E = M(1 - \dot{\xi}^2)^{-1/2} + \mathcal{V}(\xi, \dot{\xi}) \tag{3.4}$$

$$\mathcal{V}(\xi, \dot{\xi}) = \gamma \int_{-\infty}^{\infty} \mathrm{d}x\, V(x)\rho[\gamma(x - \xi)] \tag{3.5}$$

where we have used

$$\psi^\dagger\psi = \phi^\dagger L^\dagger L\phi = \gamma\phi^\dagger(1 + \dot{\xi}\sigma_y)\phi = \gamma\phi^\dagger\phi = \gamma\rho(x'). \tag{3.6}$$

The $\dot{\xi}$-dependence of $\mathcal{V}(\xi, \dot{\xi})$ arises through the γ involved on the right-hand side of (3.5). Since no time derivative of ψ appears in (3.3), we do not have to know that the t-dependence of ξ and γ in going from (3.3) to (3.4).

Let us examine $\mathcal{V}(\xi, \dot{\xi})$ of the two cases in which we are particularly interested. If

$$V(x) = -Fx \tag{3.7}$$

where F is a constant, we obtain

$$\mathcal{V}(\xi) = V(\xi) = -F\xi \tag{3.8}$$

which is independent of $\dot{\xi}$. We interpret E of (3.4) as the Hamiltonian for a point particle of mass M in classical mechanics. In terms of coordinate ξ and its conjugate momentum p_ξ, the Hamiltonian reads as

$$H = (M^2 + p_\xi^2)^{1/2} + V(\xi) \qquad p_\xi = M\dot{\xi}(1 - \dot{\xi}^2)^{-1/2}. \tag{3.9}$$

This H leads to the equation of motion

$$\dot{p}_\xi = -\mathrm{d}_\xi V(\xi) = F \tag{3.10}$$

where $\dot{p}_\xi = \mathrm{d}p_\xi/\mathrm{d}t$. This means that the soliton centroid behaves like a point particle of mass M.

One may get the impression that, for $V(x) = -Fx$ of (3.7), we have proved that the centroid of the NLD soliton exactly obeys the classical equation of motion (3.10). This is not necessarily correct because the wavefunction $\psi(x, t; \dot{\xi})$ that we have used is only an ansatz; it is not an exact solution of the NLD equation in the presence of the external potential $V(x)$. We will discuss this aspect in appendix 2.

For the harmonic oscillator potential

$$V(x) = (K/2)x^2 \tag{3.11}$$

we find

$$\mathcal{V}(\xi,\dot{\xi}) = \frac{K}{2}\gamma\int_{-\infty}^{\infty}\mathrm{d}x\,x^2\rho[\gamma(x-\xi)]$$
$$= \frac{K}{2}\gamma\int_{-\infty}^{\infty}\mathrm{d}x\,[\xi^2+(x-\xi)^2+2\xi(x-\xi)]\rho[\gamma(x-\xi)]$$
$$= V(\xi)+\frac{K}{2}(1-\dot{\xi}^2)\langle x^2\rangle \tag{3.12}$$

where

$$\langle x^2\rangle = \int_{-\infty}^{\infty}\mathrm{d}x\,x^2\rho(x). \tag{3.13}$$

This $\langle x^2\rangle$ is the mean square radius of the soliton in its rest frame; it is a constant within our ansatz. The $\dot{\xi}$-dependence of $\mathcal{V}(\xi,\dot{\xi})$ arises through the factor $\gamma^{-2}=(1-\dot{\xi}^2)$ of the last term of (3.12). This is an effect of the finite size of the soliton; it vanishes if $\langle x^2\rangle\to 0$. Such $\dot{\xi}$-dependence does not appear in the corresponding problem for the NLS equation [1, 2].

Since $\mathcal{V}(\xi,\dot{\xi})$ depends on $\dot{\xi}$, the relation between p_ξ and $\dot{\xi}$ becomes different from that of (3.9). If we start with the Lagrangian

$$L = -M(1-\dot{\xi}^2)^{1/2}-\frac{K}{2}(1+\dot{\xi}^2)\langle x^2\rangle - V(\xi) \tag{3.14}$$

we obtain

$$p_\xi = M\dot{\xi}\left[(1-\dot{\xi}^2)^{-1/2}-\frac{K}{M}\langle x^2\rangle\right] \tag{3.15}$$

$$H = p_\xi\dot{\xi}-L = M(1-\dot{\xi}^2)^{-1/2}+\mathcal{V}(\xi,\dot{\xi}). \tag{3.16}$$

This H agrees with the E of (3.4) and justifies the Lagrangian of (3.14). As a Hamiltonian, $\dot{\xi}$ in H is supposed to be eliminated in favour of p_ξ. This can be done by solving (3.15) as an equation for $\dot{\xi}$, but we do not delve into this aspect. The Lagrangian (3.14) leads to the equation of motion

$$\dot{p}_\xi = -\mathrm{d}_\xi V(\xi) = -K\xi. \tag{3.17}$$

The trajectory which this equation predicts is different from that of a point particle in potential $V(\xi)$ because of the additional term with $\langle x^2\rangle$ in p_ξ of (3.15). The acceleration of the soliton is greater than that of the corresponding point particle.

Before ending this section a remark would be in order. In our earlier study of the NLS soliton in a harmonic oscillator potential we emphasized that the external potential induces a change in the internal structure of the soliton as compared with that of the free soliton [2]. This does not happen (to the NLS soliton) if the external potential is a linear one. The NLD soliton in our ansatz has the same structure as the free soliton. This is probably a very good approximation when $V(x)$ is linear in x. When $V(x)$ is quadratic in x^2, however, the *soliton structure will change and $\langle x^2\rangle$ will become different from its free soliton counterpart.* This aspect is ignored in our ansatz. On the motion of the soliton centroid, the change in the internal structure has no effect in the NLS case. In the NLD case, there will be such an effect but we suspect that it would be much smaller than the effect of the type that we discussed in the preceding paragraph.

4. The NLD equation in the body-fixed coordinate system

The NLD equation in the presence of an external potential is given by (3.1). We have numerically solved (3.1) for $V(x)$ which is proportional to x or x^2, with appropriate initial conditions which we will explain in the next section. We have also solved (3.1) by transforming it to an equation in the 'body-fixed' coordinate system. We tried the transformation to the body-fixed coordinate system because that is how the NLS equation with an external potential has been solved [1, 2]. This time the transformation does not seem to enable us to solve the NLD equation analytically. Examining the NLD equation in two different coordinate systems, however, serves as a good consistency test.

Let us describe the transformation to the body-fixed coordinate system. We introduce a new coordinate y through

$$y = x - \xi(t) \tag{4.1}$$

where it is understood that the soliton centroid is at $x = \xi(t)$ in the 'laboratory system'. Rewrite ψ as

$$\psi(x,t) = \chi(y,t)\exp[\mathrm{i}p_\xi(t)y] \tag{4.2}$$

where $p_\xi(t)$ is the momentum of the soliton as an anticipated classical particle in the laboratory system. Equation (3.1) then becomes

$$\mathrm{i}\partial_t\chi = [-\mathrm{i}\alpha\partial_y + \beta m - g(\chi^\dagger\beta\chi)\beta + \alpha p_\xi + \mathrm{i}\dot{\xi}\partial_y]\chi + [-\dot{\xi}p_\xi + y\dot{p}_\xi + V(y+\xi)]\chi. \tag{4.3}$$

Let us consider the linear potential $V(x)$ of (3.7). It is understood that $\xi(t)$ satisfies the classical equation of motion (3.10) for a point particle of mass M. Then $\xi(t)$ and $p_\xi(t)$ are known functions of t. For solutions of the classical equation of motion, see appendix 1. The terms proportional to y cancel in the second [...] of (4.3). If we further rewrite χ as

$$\chi \to \chi\exp\left[\mathrm{i}\int^t \mathrm{d}t\,(\dot{\xi}p_\xi + F\xi)\right] = \chi\exp[\mathrm{i}(\xi p_\xi + \text{constant})] \tag{4.4}$$

(4.3) becomes

$$\mathrm{i}\partial_t\chi = [-\mathrm{i}\alpha\partial_y + \beta m - g(\chi^\dagger\beta\chi)\beta + \alpha p_\xi + \mathrm{i}\dot{\xi}\partial_y]\chi. \tag{4.5}$$

This equation determines the soliton wavefunction $\chi(y,t)$ in the body-fixed coordinate system. If the soliton behaves like a point particle, then its centroid should remain at rest in the y-coordinate system. We will see numerically that this is indeed the case in the next section.

Let us add a remark about transformation (4.1). This transformation resembles the Galilei transformation rather than the Lorentz transformation. It is not the Galilei transformation though because $\dot{\xi}$ is not a constant. One may wonder why we did not use a transformation like (3.2) which resembles the Lorentz transformation. The reason is that when speed $\dot{\xi}$ is not a constant, the inverse of such a transformation is very complicated.

5. Numerical solutions

In this section we present numerical solutions of (3.1). For the parameters of the soliton, we arbitrarily take $m = 1$ and $g = 1$ throughout. Then the rest mass of the soliton becomes $M = \ln[(3 + 5^{1/2})/2] = 0.9624$. We present seven figures. In figures 1–6 we use the potential

$$V(x) = -F(x - x_0) \tag{5.1}$$

where F is a constant. The constant x_0 which shifts the origin of the coordinate is unimportant. For a particle of rest mass M, the expected acceleration is

$$a = F/M. \tag{5.2}$$

For F, we take $F = 2.5 \times 10^{-3}$ in figures 1–6 except for figure 2 in which we take $F = 2.5 \times 10^{-3}$, 5.0×10^{-3} and 7.5×10^{-3}. Only in figure 7 do we use

$$V(x) = \tfrac{1}{2}K(x - x_0)^2 \tag{5.3}$$

where K is a constant, which we assume to be $K = 2 \times 10^{-4}$.

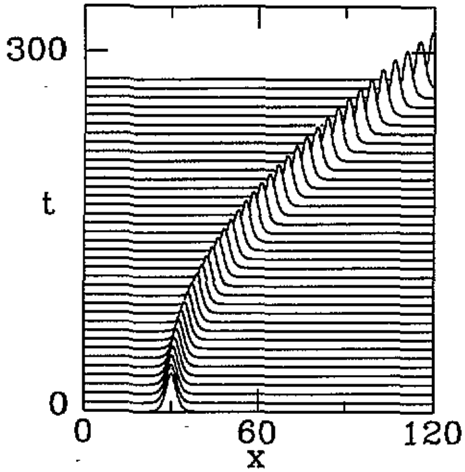

Figure 1. The trajectory of a soliton which starts at rest at $x = x_0 = 30$ and is accelerated. Its acceleration is $a = F/M$ where $F = 2.5 \times 10^{-3}$ and $M = 0.9624$. The units are such that $c = \hbar = m = 1$.

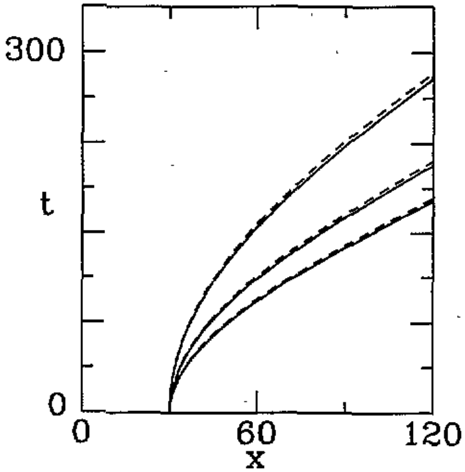

Figure 2. (a) The trajectory of the centroid (the peak of the density) of the soliton, (b) the trajectory of a classical point particle of M, and (c) the trajectory of the classical point particle of mass m (instead of M) are shown. (a) and (b), which are indistinguishable from each other, are shown by a full curve, and (c) by a broken curve. For constant force F three different values are used. $F = 2.5 \times 10^{-3}$, 5.0×10^{-3} and 7.5×10^{-3}. These values correspond to the three sets of curves, from the top to the bottom. The units are such that $c = \hbar = m = 1$.

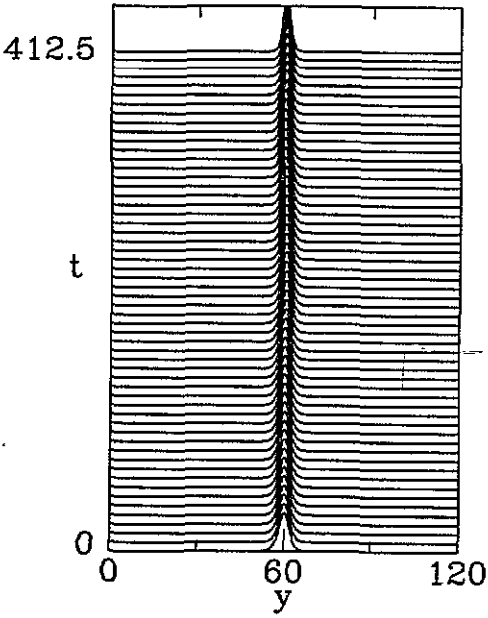

Figure 3. The soliton of figure 1 as seen in the body-fixed coordinate system. The units are such that $c = \hbar = m = 1$.

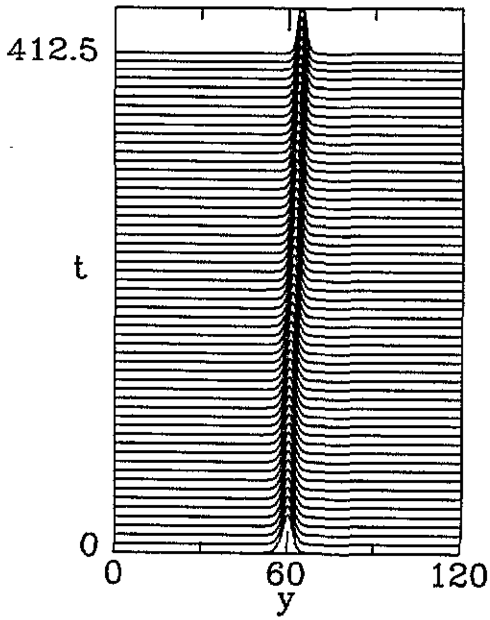

Figure 4. The same as figure 3 except that a wrong $\xi(t)$ is used as explained in section 5.

Figure 1 shows a soliton which is initially at rest at $x = x_0 = 30$. In solving (3.1) we started with $\psi(x - x_0, t = 0; v = 0)$. The speed of the soliton at $t = 300$ is $v = 0.615$ in units of c. In figure 2, for each of the three values used for F, we plot (a) the trajectory of the peak of the density of the soliton, (b) the trajectory of a point particle of mass M, and (c) the trajectory of the classical point particle of mass m (instead of M). The trajectories (a) and (b), which we find indistinguishable from each other, are shown with a full curve. The trajectory (c) is shown with a broken curve, which clearly deviates from the full curve. This means that the soliton behaves like a point particle of mass M rather than m ($> M$). For the same value of F, the acceleration is larger for M than for m. This is because $M < m$.

Figure 3 shows the soliton in the body-fixed coordinate system. Note that the soliton remains at rest at $y = 60$. Although it is difficult to see in the figure, close scrutiny reveals that the width of the soliton decreases in time. This is because of the Lorentz contraction. Recall that the soliton is being accelerated in the laboratory system. Within the accuracy of our calculation, the contraction factor agrees with $1/\gamma$.

In figure 4, we used a 'wrong' $\xi(t)$ which was obtained by assuming that the mass of the soliton is m (rather than M). The soliton does not stay at rest in this wrong body-fixed system. Instead the soliton moves in the positive y direction. In the laboratory system, the soliton moves faster than the origin of the y-coordinate system with the wrong $\xi(t)$. This is consistent with what was shown in figure 2.

In figure 5, we assumed that the potential is given by

$$V(x) = \begin{cases} 0 & \text{for } x < x_0 \\ -F(x - x_0) & \text{for } x > x_0. \end{cases} \tag{5.4}$$

This $V(x)$ is not linear in x around $x = x_0$. In solving (3.1) we let the soliton start at $x \ll x_0$ with constant speed $v = 0.1$. We set up the initial wavefunction by means of

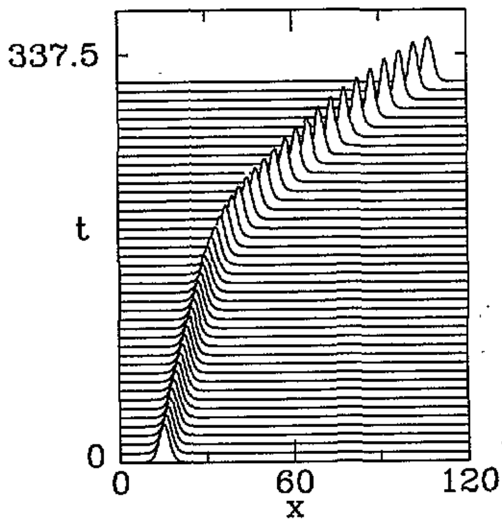

Figure 5. The same as figure 1 except that the soliton is initially moving with a constant speed of $v = 0.1$. The constant force acts in the region of $x > 30$.

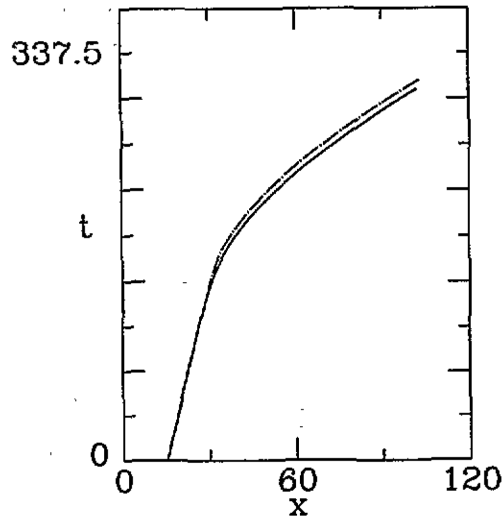

Figure 6. Comparison of the trajectory (full curve) of the soliton centroid of figure 5 with the corresponding classical trajectory (broken curve).

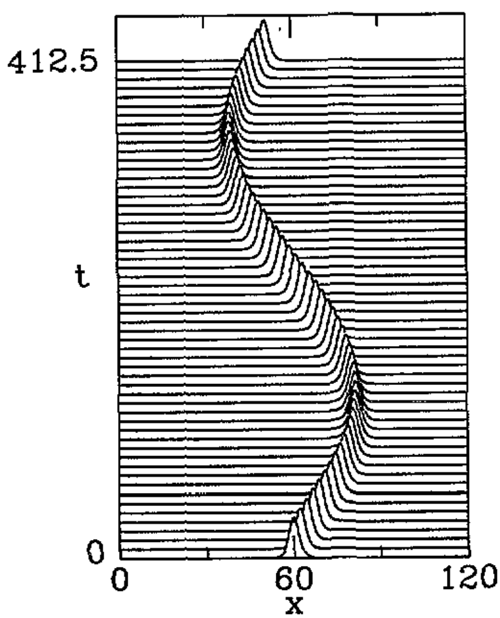

Figure 7. The soliton in the harmonic oscillator potential of (5.3) with $K = 2 \times 10^{-4}$. The units are such that $c = \hbar = m = 1$.

$\psi(x, t; v)$ in such a way that, if it were not for $V(x)$ at all, the soliton would arrive at $x = x_0 = 30$ at $t = 150$. The figure shows that the soliton begins to be accelerated around $x = x_0$.

Figure 6 compares the trajectory of the density peak of the soliton (full curve) with the corresponding point-particle trajectory (broken curve). In contrast to the situation that we saw in figure 1, these two trajectories do not agree. The soliton is ahead of the point-particle partner. At first we were puzzled by this disagreement. We realized, however, that the disagreement is due to a finite-size effect of the soliton. Because of its finite extension, the soliton begins to feel the effect of $V(x)$ of $x > x_0$ before the soliton centroid arrives at $x = x_0$. For a classical point particle the acceleration begins exactly at the moment when the particle passes $x = x_0$. Similar finite-size effects are also seen for the NLS soliton.

Figure 7 shows the soliton placed in a harmonic oscillator potential (5.3) with $K = 2 \times 10^{-4}$. The soliton starts with the initial speed $v = 0.3$ at $x = 60$. We compared the motion of the density peak of the soliton with the corresponding point particle described by (3.15) and (3.17). We found that they agree within the accuracy of our calculation. In section 3 we pointed out that (3.17) is different from the equation of motion for a point particle in potential $V(\xi)$ because the p_ξ of (3.15) is different from the usual p_ξ of (3.9). This is due to the finite size of the soliton. This is a relativistic effect. For the values of the parameters that we took, we obtain $\langle x^2 \rangle = 4.40$ and hence $(K/M)\langle x^2 \rangle = 8.80 \times 10^{-4} \ll 1$. Because of the smallness of this value of $(K/M)\langle x^2 \rangle$, the finite-size effect of the harmonic oscillator case turned out to be too small to be discernible in our calculation.

For the shape of the soliton, it shows Lorentz contraction with factor $1/\gamma$ as far as we can discern from the figure. The structure change of the type that we discussed in the last paragraph of section 4 is too small to be seen in our examples.

6. Discussion

We have examined the behaviour of the NLD soliton which is placed in an external potential $V(x)$. We assumed that the potential is either linear ($V = -Fx$) or quadratic ($V = \frac{1}{2}Kx^2$). On the basis of a collective variable ansatz, we derived the equation of motion for the NLD soliton. For the linear potential $V = -Fx$, the soliton centroid obeys the classical relativistic equation of motion (3.10). For the harmonic oscillator potential $V = \frac{1}{2}Kx^2$, we found a departure from the behaviour of the corresponding classical point particle; see (3.15). This is due to the finite size of the soliton. Next we solved the NLD equation numerically and found that the results are consistent with the prediction based on the collective variable ansatz. The departure from the classical point particle in the case of the harmonic oscillator, however, is too small to be discernible in our numerical examples. We also examined the case in which the potential is not exactly linear nor quadratic in the sense of (5.4). In that case, as shown in figure 6, we noticed a small but significant departure from the behaviour of the corresponding point particle. We interpreted this also as an effect of the finite size of the soliton. The finite-size effect of this type also shows up for the NLS soliton.

For the NLS soliton in an external potential, it is known analytically that it behaves exactly like a classical Newtonian particle if the potential is linear or quadratic in x or if the size of the soliton can be ignored [1, 2]. The results that we have found strongly suggest that similar situations hold for the NLD soliton. The NLD soliton behaves like a classical relativistic point particle if the potential is linear in x or if the size of the soliton can be ignored. There is a subtle difference between the NLS and the NLD solitons, however. For a quadratic potential the NLD soliton behaves slightly differently from its point-particle counterpart. This is a finite-size effect which has no counterpart for the NLS soliton. There is another interesting difference between the Schrödinger and Dirac cases regarding the mass of the soliton. The mass of the NLS soliton is the same as the m that appears in the wave equation whereas the mass of the NLD soliton is M of (2.7).

For the nonlinear term of the NLD equation, i.e. the 'self-interaction' term, we assumed the Lorentz scalar type. As long as the NLD equation has a free soliton solution which conforms to the Lorentz transformation (as summarized in section 2), the collective variable method of section 3 applies to any nonlinearity in the same way. Of course different nonlinear terms will lead to different values of the soliton rest mass M. Finally let us emphasize that, even for the simple potential $V(x) = -Fx$, no analytical solution of (3.1) seems to be known.

Acknowledgments

This work was supported by the Natural Sciences and Engineering Research Council of Canada and the Ministry of Education of Japan.

Appendix 1. Classical equation of motion

Consider a particle of rest mass M which is subject to a force F. The equation of motion for it is

$$\dot{p} = F \qquad p = M\gamma v \tag{A1.1}$$

where γ is the one defined by (2.10). If F is a constant, (A1.1) can be integrated with the results

$$v = \dot{\xi} = \frac{at + \gamma_0 v_0}{[1 + (at + \gamma_0 v_0)^2]^{1/2}} \tag{A1.2}$$

$$\xi = \frac{1}{a}\{[1 + (at + \gamma_0 v_0)^2]^{1/2} - \gamma_0\} \tag{A1.3}$$

where $a = F/M$ and suffix 0 refers to the value at $t = 0$. We have chosen the ξ-coordinate such that $\xi = 0$ at $t = 0$.

When the external potential is the harmonic oscillator type, the equation of motion is given by (3.17) together with (3.15). We solved (3.17) numerically and compared the solution with the soliton trajectory of figure 7.

Appendix 2. The approximate nature of solution (3.1)

In section 3 we used the ansatz

$$\psi(x, t) = L(\gamma)\phi[\gamma(x - \xi)]\exp[-\mathrm{i}\epsilon\gamma(t - \dot{\xi}x)] \tag{A2.1}$$

where $L(\gamma)$, $\phi(x)$ and ϵ are defined by (2.9), (2.6) and (2.5), respectively, but $\gamma = (1 - \dot{\xi}^2)^{-1/2}$. By substituting this into (3.1) and multiplying both sides with $L^{-1}\mathrm{e}^{\mathrm{i}\epsilon t'}$, we arrive at

$$\dot{\gamma}\left[\mathrm{i}(x - \xi)\phi'(x') + \left(\epsilon t + \frac{\mathrm{i}}{2\gamma\dot{\xi}}\sigma_y\right)\phi(x')\right] - \epsilon(\mathrm{d}_t(\gamma\dot{\xi}))x\phi(x') = V(x)\phi(x') \tag{A2.2}$$

where $\phi'(x') = \mathrm{d}\phi(x')/\mathrm{d}x'$ and $\mathrm{d}_t = \mathrm{d}/\mathrm{d}t$. In deriving (A2.2) the following formulae have been useful:

$$L^{-1} = 2^{-1/2}(\gamma+1)^{1/2}\left(1 - \frac{v\gamma}{\gamma+1}\sigma_y\right) \tag{A2.3}$$

$$L^{-1}\beta L = \gamma(1 - v\sigma_y)\beta \qquad L^{-1}\alpha L = \alpha \qquad L^{-1}\mathrm{d}_t L = (\dot{\gamma}/2v\gamma)\sigma_y. \tag{A2.4}$$

Note that $\mathrm{i}\sigma_y$ is real. Therefore all the terms in (A2.1) except for the first term in $[\ldots]$ are real. It is clear that this equation cannot be satisfied exactly. Note also that, if we assume that $\dot{\gamma} = 0$ and $V(x) = -Fx$, we obtain

$$\epsilon \mathrm{d}_t(\gamma\dot{\xi}) = F. \tag{A2.5}$$

This is the equation of motion for a particle of rest mass ϵ. Recall that $\epsilon < M$. Our numerical results show that (A2.5) is not satisfied.

References

[1] Chen H H and Liu C S 1976 *Phys. Rev. Lett.* **37** 693; 1978 *Phys. Fluids* **21** 377
de Moura M A 1988 *Phys. Rev.* A **37** 4998; 1994 *J. Phys. A: Math. Gen.* **27** 7157
[2] Nogami Y and Toyama F M 1994 *Phys. Rev.* E **49** 4497
[3] Alvarez A and Carreras B 1981 *Phys. Lett.* **86A** 327
[4] Chang S-J, Ellis S D and Lee B W 1975 *Phys. Rev.* **11** 3572
Lee S Y, Kuo T K and Gavrielides A 1975 *Phys. Rev.* D **12** 2249
Nogami Y and Toyama F M 1992 *Phys. Rev.* A **45** 5258
[5] Scharf R and Bishop A R 1992 *Phys. Rev.* A **46** R2973
[6] Boesch R and Willis C R 1990 *Phys. Rev.* B **42** 2290
Bishop A R, Campbell D K, Kumar P and Trullinger S E 1987 *Nonlinearity in Condensed Matter* (Berlin: Springer)